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Ernst Schmidt's approach to fin optimization: an extension to fins with variable conductivity and the design of ducts for fluid flow

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Abstract—In the first part of the paper, Ernst Schmidt's intuitive argument for choosing the optimum fin shape for least material is translated into analysis using the method of variational calculus. It is shown that even when the thermal conductivity of the fin is a function of temperature, the optimum shape of an individual 'heat tube' of the fin is the uniform shape, i.e. the tube the geometry of which does not vary with the longitudinal position. This generalization of Schmidt's argument is used in the search of optimum shapes for fins the materials of which have temperature-dependent conductivities. In the second part of the paper the analytical generalization of Schmidt's argument is applied to the design of ducts for fluid flow. It is shown that the shape of a duct the flow pattern or temperature of which varies with the longitudinal position can be selected optimally such that the overall flow resistance of the duct is minimized. The optimization of the duct shapes illustrated in this paper is conducted subject to one of two constraints, constant total duct volume or constant total duct wall surface.

1. INTRODUCTION

In this paper we take a new look at a great idea contributed in 1926 by Ernst Schmidt, namely, the selection of the optimum fin shape that insures the maximum heat transfer rate per total fin volume [1]. Schmidt's contribution came in the form of a short essay in which he discussed the maximization not of the total heat transfer rate but of the individual heat transfer rate through one of the many 'heat tubes' that make up the fin. He reasoned that the temperature distribution along each heat tube and along the fin should be linear, and used this conclusion in order to determine the optimum fin profile for maximum fin heat transfer per fin volume.

Schmidt's optimum fin solution was accepted on the basis of the original argument and became an integral part of the subfield of extended-surface heat transfer (see, e.g. Jakob [2] and Kern and Kraus [3]). A rigorous proof of the correctness of this solution was communicated in 1959 by Duffin [4], in whose view Schmidt's intuitive argument "is not convincing". Despite the existence of Duffin's rigorous proof, however, it is reasonable to expect that the heat transfer treatises of the future will continue to introduce Schmidt's optimum fin design conclusion based on the original intuitive argument (for more on this, see the end of Section 7).

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To aid in the future reference-level presentation of Schmidt's idea was the original objective of the present study. In it we first sought to express in compact analytical form the original argument of how a heat tube transfers heat best. This we were able to do. We also discovered that the same argument can be used in order to optimally shape a fin the thermal conductivity of which is temperature dependent. These analytical developments, which are reported in Sections 2 and 3, led us to the fluid-mechanics analog of Schmidt's design problem. In Sections 4–6 we outline a procedure for choosing the optimum shape of a duct the total volume or wall material of which are fixed, and overall fluid-flow resistance of which is to be minimized.

2. OPTIMUM TEMPERATURE DISTRIBUTION FOR FINS WITH ARBITRARY TEMPERATURE-DEPENDENT CONDUCTIVITY

The objective of this section is to express analytically the intuitive argument that was advanced in essay form by Ernst Schmidt [1]. An additional objective is to show that Schmidt's argument holds not only for materials with constant thermal conductivity—the existence of which is assumed routinely in extended-surface analysis—but also for materials with an arbitrary relationship between conductivity and temperature, $k(T)$.

Consider the unidirectional conduction through the slender solid element ('heat tube' in Ernst Schmidt's terminology) sketched in Fig. 1. The cross-sectional area normal to the heat current q is an unspecified

NOMENCLATURE

a coefficient, k/T
A cross-sectional area of the heat tube, Fig. 1
A cross-sectional area of the duct, equations (28) and (29)
$c_1, \ldots, c_6$ constants
c_p specific heat at constant pressure
C_1, C_2, C_3 constants
D diameter
D_h hydraulic diameter
f friction factor
I, I_1, I_2, I_3, I_4 integrals
k thermal conductivity
L length
$\dot{m}$ mass flow rate
M wall material (surface) constraint, equation (24)
Nu Nusselt number, equation (45)
p wetted perimeter of the fin cross-section
p_d wetted perimeter of the duct cross-section
P pressure
q heat transfer rate
R duct flow resistance
Re Reynolds number
S cross-sectional area of the fin, Fig. 2
T absolute temperature
T_f ambient fluid temperature
ΔT temperature difference
U fluid velocity averaged over the duct cross-section
V volume constraint, equation (1)
W width of the flat rectangular cross-section of a duct
x longitudinal coordinate.

Greek symbols
θ thermal boundary potential, equation (5)
$\lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ Lagrange multipliers
ν kinematic viscosity
ρ density
τ dimensionless temperature ratio, T_0/T_f
τ_w wall shear stress
ϕ function of longitudinal position
ψ shorthand notation for $\ln(T_L/T_0)$.

Subscripts
$(\ )_{\text{constant-}D}$ associated with a round tube of constant diameter
$(\ )_k$ associated with constant thermal conductivity
$(\ )_L$ property at $x = L$
$(\ )_{\min}$ associated with a round-cross-section duct of optimum diameter
$(\ )_{\text{opt}}$ optimum
$(\ )_0$ property at $x = 0$.

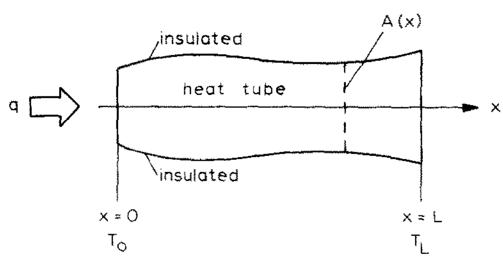

FIG. 1. One-dimensional conduction through a heat tube of fixed length and volume.

function of longitudinal position, $A(x)$. The heat current is driven by the temperature difference $(T_0 - T_L)$, which is applied between the $x = 0$ and L ends of the heat tube. The length of the heat tube (L) and its total volume (V) are fixed

$$\int_0^L A(x)\,\mathrm{d}x = V, \text{ constant.} \tag{1}$$

Schmidt argued that the optimum heat tube shape that maximizes the heat current through a heat tube of fixed volume and length is the 'uniform' shape, which in the present terminology is written as

$$A_{\text{opt}} = \frac{V}{L}, \text{ constant.} \tag{2}$$

We can demonstrate the validity of this result analytically by first writing

$$q = -k(T)A(x)\frac{\mathrm{d}T}{\mathrm{d}x} \tag{3}$$

and noting that q is not a function of x. Integrating equation (3) over the entire path traveled by q, we obtain

$$\int_0^L \frac{\mathrm{d}x}{A(x)} = \frac{1}{q}\int_{T_L}^{T_0} k(T)\,\mathrm{d}T. \tag{4}$$

The thermal conductivity integral appearing on the right-hand side may be rewritten using Garwin's 'thermal boundary potential' function [5]

$$\theta(T) = \int_0^T k(T')\,\mathrm{d}T' \tag{5}$$

and the result is

$$\int_0^L \frac{\mathrm{d}x}{A(x)} = \frac{\theta(T_0) - \theta(T_L)}{q}. \tag{6}$$

The physical meaning of this result is that the heat current q is driven by the thermal boundary potential difference $\theta(T_0)-\theta(T_L)$ across a thermal resistance of size

$$\int_0^L \mathrm{d}x/A.$$

The analytical counterpart to Ernst Schmidt's argument is the variational calculus problem consisting of finding the optimum function $A(x)$ that minimizes the thermal resistance integral of equation (6) subject to volume integral (1). The problem is equivalent to minimizing the aggregate integral

$$I=\int_0^L\left(\frac{1}{A}+\lambda A\right)\mathrm{d}x \tag{7}$$

subject to no constraints [6]. Note in the integrand of this new integral the Lagrange multiplier λ and the integrands of the original volume constraint (1) and thermal resistance integral (6). The variational calculus solution to minimizing I is straightforward

$$A_{\mathrm{opt}}=\lambda^{-1/2}, \text{ constant} \tag{8}$$

for which the constant $\lambda^{-1/2}$ is determined by substituting equation (8) into volume constraint (1). The final form of the solution obtained in this manner was given already in equation (2).

The implications of this result in the design of extended surfaces have been exploited already by Schmidt and his successors [1–3]. The design of fins of optimum profile (where 'optimum' means maximum total fin heat transfer subject to fixed fin volume) begins with viewing the fin as a bundle of heat tubes of the type analyzed in Fig. 1, and seeing that the temperature distribution along each tube must be linear, provided k = constant (compare equations (2) and (3)). The optimum fin profile emerges after substituting the constant-$\mathrm{d}T/\mathrm{d}x$ conclusion into the differential energy equation for the fin.

The design conclusions associated with the preceding analysis are more general, as the variational calculus solution (2) applies to a conducting heat tube of arbitrary conductivity, $k(T)$. Optimum fin profiles may be pursued in the same manner as in the post-Schmidt work, that is, by combining the differential energy equation of the extended surface with the conclusion

$$k(T)\frac{\mathrm{d}T}{\mathrm{d}x}=\text{constant} \tag{9}$$

where k is now a known function of T. The above conclusion follows from equations (2) and (3). The optimum temperature distribution along the heat tube of Fig. 1 (and along the fins considered later) is obtained by solving equation (9) for the unknown function $T(x)$.

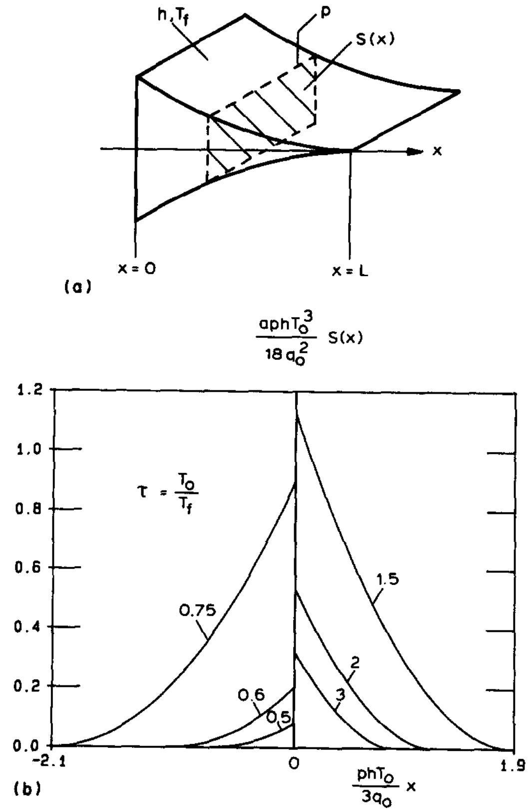

FIG. 2. Optimum profiles for a two-dimensional fin the thermal conductivity of which is proportional to the absolute temperature.

3. OPTIMUM FIN SHAPES (PROFILES) FOR FINS WITH VARIABLE CONDUCTIVITY

As an example consider the design of copper fins for heat exchangers operating at low temperatures, e.g. in the range between the normal boiling points of helium and hydrogen. At such temperatures the conductivity of copper of high purity is proportional to the absolute temperature [7]. In this case if we integrate equation (9) we obtain a longitudinal temperature distribution of the type

$$T=(c_1x+c_2)^{1/2}. \tag{10}$$

The energy equation for a fin of arbitrary cross-section $S(x)$ and wetted perimeter $p(x)$ exposed to an external fluid (T_f) across a constant heat transfer coefficient (h) is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k(T)S(x)\frac{\mathrm{d}T}{\mathrm{d}x}\right]-p(x)h(T-T_f)=0. \tag{11}$$

In the first part of this example we consider a fin of two-dimensional geometry (Fig. 2), the wetted perimeter p of which is a constant. Combining equations (9)–(11) we obtain

$$\frac{dS}{dx} = \frac{ph}{k(T)\dfrac{dT}{dx}}[(c_1x+c_2)^{1/2}-T_f] \quad (12)$$

which yields the following solution for the optimum fin profile $S(x)$ of Fig. 2:

$$S = \frac{ph}{k(T)\dfrac{dT}{dx}}\left[\frac{2}{3c_1}(c_1x+c_2)^{3/2}-T_fx\right]+c_3. \quad (13)$$

The three constants (c_1, c_2, c_3) are determined from the following three conditions:

(a) the base temperature condition $T = T_0$ at $x = 0$, which yields

$$c_2 = T_0^2; \quad (14)$$

(b) the heat flux condition $q = 0$ at $x = L$, which yields

$$c_3 = -\frac{ph}{k(T)\dfrac{dT}{dx}}\left[\frac{2}{3c_1}(c_1L+c_2)^{3/2}-T_fL\right]; \quad (15)$$

(c) the condition of minimum fin volume $dV/dL = 0$, which yields

$$c_1 = \frac{T_f^2-T_0^2}{L}. \quad (16)$$

Since the thermal conductivity is proportional to the absolute temperature, $k(T) = aT$, the resulting solution $S(x)$ can be cast in the form

$$S = \frac{2ph}{aT_0}\frac{L^2\tau}{1-\tau^2}\left[1-\frac{x}{L}+\frac{2/3}{1-\tau^2}\times\left\{\left[(1-\tau^2)\frac{x}{L}+\tau^2\right]^{3/2}-1\right\}\right] \quad (17)$$

where $\tau = T_0/T_f$. If, instead of specifying a fin length, L, we fix the base heat flux ($q = q_0$ at $x = 0$) then L can be expressed as

$$L = \frac{3q_0}{phT_0}\frac{\tau(1+\tau)}{(\tau-1)(2\tau+1)}. \quad (18)$$

Figure 2 shows a sample of fin profiles based on equations (17) and (18) in dimensionless form. The abscissa is negative for $\tau < 1$ because q_0 is negative. Fins become increasingly more slender and shorter if τ (>1) increases or τ (<1) decreases. In the special limit $\tau \to 1$ both L and $S(x = 0)$ tend to infinity. This feature is present also in Schmidt's solution [1], which in the present nomenclature reads

$$S_k = \frac{ph}{2k}L_k^2\left(\frac{x}{L_k}-1\right)^2 \quad (19)$$

$$L_k = \frac{2q_0}{phT_0}\frac{\tau}{\tau-1}. \quad (20)$$

Subscript k refers to the assumption of constant conductivity in Schmidt's solution.

Table 1. Comparison of the optimum fin lengths and base cross-sections calculated for fins with variable conductivity (L, S) and fins with constant conductivity (L_k, S_k)

τ	$\frac{L}{L_k}$	$\left.\frac{S(x=0)}{S_k(x=0)}\right\|_{I}$	$\left.\frac{S(x=0)}{S_k(x=0)}\right\|_{II}$	$\left.\frac{S(x=0)}{S_k(x=0)}\right\|_{III}$
0.5	1.13	0.75	1.13	1.50
0.6	1.09	0.82	1.09	1.36
0.75	1.05	0.90	1.05	1.20
1	1.00	1.00	1.00	1.00
1.5	0.94	1.13	0.94	0.75
2	0.90	1.20	0.90	0.60
3	0.86	1.29	0.86	0.43

In order to compare the present profiles with Schmidt's solution we must first select a method for estimating the effective 'constant k' of a fin the conductivity of which actually varies as $k = aT$. Three possible rules are presented below

$$\text{I.} \quad k = aT_0$$

$$\text{II.} \quad k = a\frac{T_0+T_f}{2}$$

$$\text{III.} \quad k = aT_f. \quad (21)$$

Table 1 shows the relative size of the fin lengths and base cross-sections suggested by Schmidt's and the present solution. The numerical values listed in the table have been rounded off to no more than two decimal places. When $\tau = 1$ the two solutions are in perfect agreement. When $\tau \neq 1$ the optimum constant-k fin design depends on the estimated value of k, especially for large and small τ's. In each case, the second method (see II in equation (21)) proves to be the best way to evaluate the effective constant conductivity.

In the second part of this example we consider the design of a spine the geometry of which at any x is dictated solely by the local diameter $D(x)$. Noting that S is proportional to D^2 and that $p \sim D$, we combine equation (11) with equations (9) and (10) to obtain an equation for $D(x)$

$$\frac{dD}{dx} = \frac{2h}{k(T)\dfrac{dT}{dx}}[(c_1x+c_2)^{1/2}-T_f]. \quad (22)$$

This equation is of the same type as in the case of two-dimensional fins, equation (12). Solving equation (22) and invoking the same three conditions as in the preceding example, we obtain the optimum diameter for a spine the conductivity of which is proportional to the absolute temperature

$$D = \frac{4h}{aT_0}\frac{L^2\tau}{1-\tau^2}\left[1-\frac{x}{L}+\frac{2/3}{1-\tau^2}\times\left\{\left[(1-\tau^2)\frac{x}{L}+\tau^2\right]^{3/2}-1\right\}\right]. \quad (23)$$

The discussion of equation (23) is analogous to that of equation (17). It is worth mentioning that the shapes $D(x)$ and $2S(x)/p$ are identical if L is fixed.

4. THE DUCT FLOW PROBLEM: OPTIMUM DUCT GEOMETRY FOR MINIMUM FLOW RESISTANCE

In this and the remaining sections of the paper we exploit the fluid mechanics implications of the heat tube analysis centered around Fig. 1. In brief, that analysis showed that when the total volume of the heat tube is constrained there exists an optimum heat tube geometry that minimizes the overall thermal resistance across the tube. We have reasons to expect the existence of an equivalent optimum geometry in the design of an actual duct for the flow of a stream ($\dot{m}$) between two pressure levels (P_0 and P_L) separated by a fixed distance L.

In the search for optimum duct geometries there are at least *two* constraints to consider, first, the duct volume constraint (which is written exactly the same way as in equation (1)), and second, the duct wall material constraint

$$\int_0^L p_d(x)\,dx = M \tag{24}$$

where $p_d(x)$ is the wetted perimeter of the duct cross-section $A(x)$. Volume constraint (1) is most relevant in the design of compact heat exchangers, where the volume occupied by the heat exchanger is an important design constraint (e.g. heat exchangers for the power plants of navy ships and submarines). Duct material constraint (24), on the other hand, is crucial in designs where the cost of the duct material is high (e.g. high-purity copper) or where the weight of the overall heat exchanger is constrained (e.g. heat exchangers for airborne and space applications).

It remains to show that the flow resistance through a duct of fixed length (L) and overall pressure drop ($P_0 - P_L$) is given by a geometry-dependent integral similar to the one encountered in the case of a heat tube, equation (6). Regardless of whether the duct flow is laminar, turbulent or fully developed, the local pressure gradient along the duct is

$$\frac{dP}{dx} = -\tau_w(x)\frac{p_d(x)}{A(x)} \tag{25}$$

where τ_w is the frictional shear stress averaged over the wetted perimeter p_d. Using the usual definitions for friction factor, Reynolds number, mass flow rate and hydraulic diameter

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{26}$$

$$Re = \frac{D_h U}{\nu} \tag{27}$$

$$\dot{m} = \rho U A \tag{28}$$

$$D_h = \frac{4A}{p_d} \tag{29}$$

in which U is the velocity averaged over the duct cross-section, pressure gradient formula (25) can be integrated from $x = 0$ to L. What results is an integral expression for the overall duct-flow resistance

$$\frac{P_0 - P_L}{\dot{m}} = \int_0^L \frac{\nu}{2} f\,Re\,\frac{p_d}{A^2 D_h}\,dx \tag{30}$$

which is the duct-flow equivalent of thermal resistance formula (6).

The integrand of the above integral is in general a function of x. This dependence follows directly from the x-dependent geometry of the duct (p_d, A, D_h) and from the possible longitudinal development of the flow (as shown in the next section, in the entrance region of a duct the product $f\,Re$ decreases along x). The integrand of the integral in equation (30) depends on x also indirectly, via the temperature-dependent property ν: if the duct is part of a heat exchanger the bulk temperature of the fluid varies with x, and so does ν. In what follows we consider all these possibilities separately as we seek the minimization of flow resistance integral (30) subject to either fixed volume (1) or fixed duct wall surface (24).

5. THE OPTIMUM SHAPE OF HYDRODYNAMIC DUCT ENTRANCE REGIONS

Consider as a first example the hydraulic entrance region to a duct of round cross-section (diameter $D(x)$) through which the flow is isothermal such that ν may be regarded as constant. Note further that since the cross-section is circular D_h is equal to D, while $p_d = \pi D$ and $A = \pi D^2/4$. It follows that the geometric group $p_d/(A^2 D_h)$ appearing in equation (30) scales as D^{-4}.

In the developing entrance region of a tube of constant diameter the group $f\,Re$ decreases approximately as $x^{-1/2}$. This group is usually plotted in dimensionless terms vs $x/(D\,Re)$ (see, e.g. ref. [8]). In the present problem D is not a constant—in fact, to find the optimum function $D(x)$ is the object of the analysis. Consequently, instead of $f\,Re$ we write $\phi(x)$, where the function ϕ could be determined in principle after the duct shape $D(x)$ is known. If the duct shape $D(x)$ turns out to be a weak function of x, that is, if the duct diameter is nearly constant throughout the $0 < x < L$ domain, then it is reasonable to expect ϕ to decrease as $x^{-1/2}$. We return to this observation at the end of this section.

As a summary to the preceding two paragraphs we conclude that the integrand of equation (30) behaves as ϕ/D^4. On the other hand, the integrand of volume constraint (1) varies as D^2. The problem of finding the function $D(x)$ that minimizes the flow resistance

(30) subject to the fixed volume (1) reduces to minimizing the integral

$$I_1 = \int_0^L \left(\frac{\phi}{D^4} + \lambda_1 D^2 \right) \mathrm{d}x \tag{31}$$

where λ_1 is another Lagrange multiplier. The solution has the form

$$D_{\text{opt}}(x) = \left[\frac{2}{\lambda_1} \phi(x) \right]^{1/6} \tag{32}$$

where, assuming that $\phi(x)$ is known, the constant λ_1 is determined by substituting equation (32) into volume constraint (1).

We develop a better feel for the optimum duct shape prescribed by equation (32) by *assuming* that D_{opt} is a sufficiently weak function of x so that we may take $x^{-1/2}$ as the x-dependence of the function ϕ. Combining this assumption with equation (32) we conclude that D_{opt} must vary approximately as $x^{-1/12}$, which is indeed a weak function of longitudinal position. It means that equation (32) reads approximately

$$D_{\text{opt}}(x) \cong (\text{constant}) x^{-1/12} \tag{33}$$

in other words, for minimum flow resistance the entrance region to a pipe should be shaped like a very long trumpet. The constant listed in equation (33) is easily determined from volume constraint (1), so that the closed-form conclusion of this first example is

$$D_{\text{opt}}(x) \cong \left(\frac{10V}{3\pi L} \right)^{1/2} \left(\frac{L}{x} \right)^{1/12} \tag{34}$$

A similar conclusion is reached if the flow resistance of the same entrance region is minimized subject to the wall surface (material) constraint (24). The start of the variational calculus problem is the aggregate integral

$$I_2 = \int_0^L \left(\frac{\phi}{D^4} + \lambda_2 D \right) \mathrm{d}x \tag{35}$$

and, following the same logic that led to equation (33), the end result is the optimum entrance region shape

$$D_{\text{opt}}(x) \cong \frac{9M}{10\pi L} \left(\frac{L}{x} \right)^{1/10}. \tag{36}$$

The entrance regions of other duct geometries may be optimized in the same way. If the duct cross-section is a flat rectangle of height $D(x)$ and constant width W, such that $W \gg D(x)$, the group $p_d/(A^2 D_h)$ of equation (30) varies as $1/(D^3 W)$. Assuming that the x-dependence of D is sufficiently weak, the $f\,Re$ product may be recognized again as a function proportional to $x^{-1/2}$. The optimum $D(x)$ function that minimizes the flow resistance integral (30) subject to volume constraint (1) turns out to be

$$D_{\text{opt}}(x) \cong \frac{7V}{8WL} \left(\frac{L}{x} \right)^{1/8}. \tag{37}$$

The optimum entrance shape is such that the 'plate-to-plate' spacing D tapers down very slowly as the flow develops downstream.

If instead of the volume we keep the total duct wall surface fixed

$$\int_0^L 2W \, \mathrm{d}x = M \tag{38}$$

we find first that the M-constraint does not depend on the narrow spacing of the duct cross-section, D. This means that an optimum wall-to-wall spacing function $D(x)$ does not exist. Instead, we may consider D as constant and the cross-section width W variable, while D remains negligible with respect to $W(x)$. The optimum entrance width that minimizes the overall flow resistance subject to constraint (38) is

$$W_{\text{opt}}(x) \cong \frac{3M}{8L} \left(\frac{L}{x} \right)^{1/4}. \tag{39}$$

This optimum entrance geometry is one in which the width of the flat cross-section decreases gradually in the flow direction, while D remains constant.

6. THE OPTIMUM SHAPE OF DUCTS WITH LONGITUDINAL TEMPERATURE VARIATION

The examples treated in the preceding section were all based on the assumption that the duct is isothermal and the viscosity ν is constant. The existence of optimum duct shapes was traced to the 'developing' nature of the flow, that is, to the x-dependence of the group $f\,Re$ in the entrance region. In the current section we consider the reverse situation and assume the fully developed laminar flow through a duct the longitudinal temperature distribution $T(x)$ of which is known. Examples of heat exchangers in which $T(x)$ does not depend on the hydraulic design of the flow passage are presented later in this section.

For the sake of simplicity we consider a circular-cross-section duct of diameter $D(x)$. The duct is long and slender enough so that the flow may be treated as fully developed with constant $f\,Re$ at any longitudinal location x (note that for a round cross-section $f\,Re = 16$). We further assume that the fluid is an ideal gas and note that at constant pressure the kinematic viscosity of this fluid increases as $T^{1.7}$ [7].†

In summary, the integrand of the flow resistance integral (30) varies as $T^{1.7}/D^4$, where $T(x)$ is known and $D(x)$ is to be selected optimally. If this selection

† A pressure drop exists along any duct with fluid flowing through it. The 'constant pressure' assumption made here means that the pressure drop $(P_0 - P_L)$ is negligible compared with the absolute pressure level P_0.

is subjected to volume constraint (1), the problem reduces to minimizing

$$I_3 = \int_0^L \left(\frac{T^{1.7}}{D^4} + \lambda_3 D^2\right) dx \tag{40}$$

and the solution is

$$D_{opt}(x) = (\text{constant}) T^{1.7/6}(x). \tag{41}$$

The constant coefficient is easily determined from volume integral (1). On the other hand, if the search for $D_{opt}(x)$ is subjected to fixed duct surface or material, equation (24), the analysis begins with minimizing

$$I_4 = \int_0^L \left(\frac{T^{1.7}}{D^4} + \lambda_4 D\right) dx \tag{42}$$

and ends with

$$D_{opt}(x) = (\text{constant}) T^{1.7/5}(x) \tag{43}$$

where the 'constant' coefficient is different from the coefficients noted already in equations (41) and (33).

The combined message of the results of equations (41) and (43) is that when the absolute temperature varies significantly along the duct the optimum duct diameter also varies noticeably between $x = 0$ and L. This variation is such that the duct opens up (D increases) toward the warm temperature end of the duct. These conclusions invite us to think first of cryogenic engineering applications in which heat exchangers span large intervals on the absolute temperature scale. One specific example is the main counterflow heat exchanger of a helium liquefier. The low pressure stream of the counterflow flows through the duct analyzed already in this section (length L, diameter D, temperature distribution $T(x)$). The high pressure stream flows toward low temperatures and, at any x, its temperature exceeds the temperature of the low pressure stream by the temperature difference ΔT. It is easy to prove that ΔT is constant (i.e. x independent) when the counterflow is balanced, that is, when the capacity rate $\dot{m}c_p$ is the same for both streams.

Of interest is the longitudinal $T(x)$ distribution dictated by the counterflow heat exchanger arrangement described above. Assuming that the stream-to-stream temperature difference is dominated by the temperature difference between the low pressure stream and the duct wall that surrounds it, the first law for the low pressure stream may be written as

$$\dot{m}c_p \, dT = h p_d \Delta T \, dx. \tag{44}$$

The heat transfer coefficient is in this case

$$h = \frac{k}{D} Nu \tag{45}$$

where Nu is a constant (recall the constant-$f\,Re$ assumption made earlier) and where the thermal conductivity k increases as $T^{0.7}$ [7]. The integral of equation (44) reads

$$\frac{\dot{m}c_p}{Nu\Delta T} \int_{T_0}^{T} \frac{dT'}{k(T')} = \int_0^x \frac{p_d}{D} dx'. \tag{46}$$

On the right-hand side of equation (46) we note the constant integrand $p_d(x)/D(x) = \pi$. In view of everything else mentioned until now, equation (46) suggests a duct temperature distribution of the form

$$T(x) = T_0(1 + c_4 x)^{1/0.3}. \tag{47}$$

The value of the constant c_4 can be identified by carrying out the two integrals appearing in equation (46).

In conclusion, combining equation (47) with the earlier results (41) and (43) we learn that the optimum tube shape at constant total volume is

$$D_{opt}(x) = c_5(1 + c_4 x)^{0.94} \tag{48}$$

and at constant total duct surface

$$D_{opt}(x) = c_6(1 + c_4 x)^{1.13}. \tag{49}$$

The new constants c_5 and c_6 follow again from the respective integral constraints, equations (1) and (24). These constants should not be confused with the constants used in the optimum fin solutions of Section 3.

An even simpler example of a cryogenic heat exchanger with prescribed longitudinal distribution is the single-stream cooling effect required by various features of thermal insulation (mechanical supports, radiation shields, electrical cables, etc. [9]). The distribution of temperature along the thermal insulation between T_0 and T_L (and along the duct that guides the coolant $\dot{m}$) is determined purely thermodynamically by minimizing the heat transfer irreversibility of the heat leak path provided by the insulation.

If, for example, the 'insulation' is a long mechanical support with constant thermal conductivity, the thermodynamically optimum distribution $T(x)$ along the support is

$$T(x) = T_0 \exp\left[\frac{x}{L} \ln \frac{T_L}{T_0}\right]. \tag{50}$$

This temperature distribution can be maintained by placing the conduction heat leak of the support in counterflow with a stream with constant capacity rate $\dot{m}c_p$ (e.g. a single stream, $\dot{m}$, containing an ideal gas with constant specific heat). The $T(x)$ distribution of equation (50) is considerably more general: it represents also the optimum cooling required by a constant-k electrical power cable that stretches from T_0 to T_L [10]. Note that in this application the prescribed $T(x)$ distribution (50) is independent of the electrical current level and Joulean heating in the cable.

It is a simple matter to determine the coolant passage geometry $D(x)$ that minimizes the overall fluid flow resistance in the heat exchanger required for maintaining equation (50). Combining equations (41) and (50) we find that the optimum duct diameter for a duct with fixed total volume (V) is

$$D_{\text{opt}}(x) = \left[\frac{4V}{\pi L}\frac{(1.7/3)\ln(T_L/T_0)}{(T_L/T_0)^{1.7/3}-1}\right]^{1/2} \times \exp\left[(1.7/6)\frac{x}{L}\ln\frac{T_L}{T_0}\right]. \quad (51)$$

Likewise, equations (43) and (50) deliver the optimum flow passage geometry for a minimum duct wall surface

$$D_{\text{opt}}(x) = \frac{M}{\pi L}\frac{(1.7/5)\ln(T_L/T_0)}{(T_L/T_0)^{1.7/5}-1} \times \exp\left[(1.7/5)\frac{x}{L}\ln\frac{T_L}{T_0}\right]. \quad (52)$$

In either case the optimum passage geometry is one in which the tube diameter increases toward the warm end of the heat exchanger. This effect is particularly noticeable in ducts spanning high T_L/T_0 ratios, as is illustrated in the next section.

7. CONCLUDING REMARKS

Starting out with the analytical rewording of Ernst Schmidt's original argument we developed first an extension of his fin design method to variable-conductivity fin materials. We then constructed the 'duct flow' analog of Schmidt's 'heat tube' and, on the basis of examples, we illustrated the search for optimum duct geometries that minimize the overall fluid flow resistance posed by the duct.

If the resulting optimum duct size is only a weak function of longitudinal position (e.g. equation (34)), we can expect a very small difference between the minimum flow resistance secured by the optimum design and the resistance associated with the much simpler design in which the duct size is independent of x. On the other hand, if the optimum duct size depends strongly on x the optimum design method presented here yields significant savings in pumping power relative to the uniform-duct-size design. The way in which the value of this design methodology varies from one application to the next can be illustrated by considering the last example, for which the optimum duct diameter for fixed total volume is given in equation (51). The minimum flow resistance (30) associated with this optimum duct geometry is

$$R_{\min} = \left(\frac{P_0-P_L}{\dot{m}}\right)_{D=D_{\text{opt}}} = \int_0^L \frac{v}{2} f\, Re \frac{p_d}{A^2 D_h}\,dx \quad (53)$$

in which

$$\frac{v}{2} = C_1 T^{1.7} \quad (54)$$

$$f\,Re = \text{constant} \quad (55)$$

$$\frac{p_d}{A^2 D_h} = \frac{16}{\pi} D^{-4} \quad (56)$$

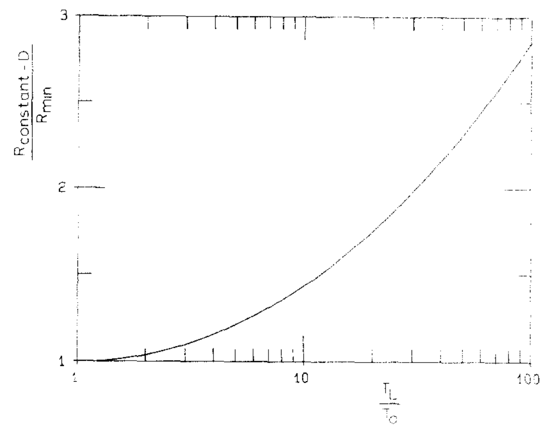

FIG. 3. The relative savings in flow resistance associated with the optimum shaping of the flow passages of cryogenic heat exchangers.

$$T = T_0 \exp\left(\psi\frac{x}{L}\right), \quad \psi = \ln\frac{T_L}{T_0}. \quad (57)$$

Abbreviating equation (51) as

$$D_{\text{opt}} = C_3 \exp\left[(1.7/6)\psi\frac{x}{L}\right] \quad (51')$$

integral (53) leads eventually to

$$R_{\min} = \frac{C_2 L}{(1.7/3)\psi C_3^4}\{\exp[(1.7/3)\psi]-1\} \quad (58)$$

where

$$C_2 = C_1 T_0^{1.7} f\,Re\frac{16}{\pi}. \quad (59)$$

The minimum flow resistance estimated above can be compared with the resistance of a duct the 'constant-D' geometry of which satisfies the same total volume constraint

$$R_{\text{constant-}D} = \left(\frac{P_0-P_L}{\dot{m}}\right)_{D=\text{constant}} \quad (60)$$

Omitting the algebra, the final result can be expressed in terms of the quantities defined in the preceding paragraph

$$R_{\text{constant-}D} = \frac{C_2}{C_3^4}L\frac{\exp(1.7\psi)-1}{1.7\psi}\left[\frac{\exp[(1.7/3)\psi]-1}{(1.7/3)\psi}\right]^{-2} \quad (61)$$

so that the ratio of the two resistances becomes a function of only the temperature ratio T_L/T_0 (note the shorthand notation $\psi = \ln T_L/T_0$)

$$\frac{R_{\text{constant-}D}}{R_{\min}} = \frac{\exp(1.7\psi)-1}{1.7\psi}\left[\frac{\exp[(1.7/3)\psi]-1}{(1.7/3)\psi}\right]^{-3}. \quad (62)$$

Figure 3 shows the way in which $R_{\min}$ and $R_{\text{constant-}D}$ compare as T_L/T_0 increases. The minimum resistance associated with the optimum design decreases dramatically towards the cryogenic engineering end of the T_L/T_0 scale, that is, when $T_L/T_0 > 10$. In that range, the R ratio (62) increases asymptotically as 0.11 $(\ln T_L/T_0)^2$.

In conclusion, the optimum shaping of the flow passages of cryogenic heat exchangers (high T_L/T_0 ratios) promises significant savings. Of course, nobody is suggesting the manufacture of trumpet-shaped tubes according to equations (51) and (52). Nevertheless, savings comparable to those predicted in Fig. 3 can be achieved by constructing a heat exchanger passage the size of which varies in steps, in a way that mimicks the smooth $D(x)$ function obtained analytically. The optimization of stepped-size heat exchanger passages can be pursued in the same manner as in Sections 4–6 of this paper: the only difference will be that the variational calculus problem will be replaced by the problem of solving a system of equations for all the unknown size steps of the duct.

We use this opportunity to draw attention to the fact that Schmidt's and Duffin's conclusion regarding the optimum shaping of fins has come under criticism, beginning with a 1974 paper by Maday [11]. A very interesting overview of the post-1974 debate was presented by Snider and Kraus [12]. These authors point out that Schmidt and Duffin neglected the so-called 'length of arc', i.e. the additional contribution made to the lateral contact area by the slope (deviation from the straight shape, parallel to the x-axis in Fig. 1) of the edge of the longitudinal cut made through the fin. It turns out that the optimum fin differs from Schmidt's if the length of arc is taken into account in the design of fins that are not necessarily slender.

In light of this, our paper represents a simple analytical alternative to Schmidt's approach and, in addition, an extension of that approach to both fins and duct flows. Our work does not contribute to the post-1974 debate and, in this paper, at least, does not champion the supremacy of Schmidt's design over the more recent developments.

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REFERENCES

1. E. Schmidt, Die Wärmeübertragung durch Rippen, *Z. Ver. Dt. Ing.* **70,** 885–889, 947–951 (1926).
2. M. Jakob, *Heat Transfer*, Vol. 1, p. 219. Wiley, New York (1950).
3. D. Q. Kern and A. D. Kraus, *Extended Surface Heat Transfer*, p. 98. McGraw-Hill, New York (1972).
4. R. J. Duffin, A variational problem relating to cooling fins, *J. Math. Mech.* **8,** 47–56 (1959).
5. R. L. Garwin, Calculation of heat flow in a medium the conductivity of which varies with temperature, *Rev. Scient. Instrum.* **27,** 826–828 (1956).
6. F. B. Hildebrand, *Advanced Calculus for Applications*, p. 355. Prentice-Hall, Englewood Cliffs, New Jersey (1962).
7. U. Grigull and H. Sandner, *Heat Conduction* (Translated by J. Kestin), p. 8. Hemisphere, Washington, DC (1984).
8. A. Bejan, *Convection Heat Transfer*, p. 71. Wiley, New York (1984).
9. A. Bejan, *Entropy Generation through Heat and Fluid Flow*, Chaps 9 and 10. Wiley, New York (1982).
10. R. Agsten, Thermodynamic optimization of current leads into low temperature regions, *Cryogenics* **13,** 141–146 (1973).
11. C. J. Maday, The minimum weight one-dimensional straight cooling fin, *J. Engng Ind.* **96,** 161–165 (1974).
12. A. D. Snider and A. D. Kraus, The quest for the optimum longitudinal fin profile, *Thermal/Mechanical Heat Exchanger Design—Karl Gardner Memorial Session* (Edited by K. P. Singh and S. M. Shenkman), ASME HTD Vol. 64, pp. 43–48 (1986).

APPROCHE D'ERNST SCHMIDT POUR L'OPTIMISATION D'AILETTE: EXTENSION AUX AILETTES AVEC CONDUCTIVITE VARIABLE ET CONCEPTION DE CANAUX POUR ECOULEMENT DE FLUIDE

Résumé—Dans la première partie, l'argument intuitif de E. Schmidt pour choisir la forme optimale à moindre matière est traduite dans une analyse utilisant la méthode du calcul variationnel. Même lorsque la conductivité thermique de l'ailette est fonction de la température la forme optimale d'un "tube individuel de courant" dans l'ailette est celle uniforme, c'est-à-dire le tube dont la géométrie ne varie pas avec la position longitudinale. Cette généralisation de l'argument de Schmidt est utilisée dans la recherche de formes optimales des ailettes dont le matériau a une conductivité variable avec la température. Dans la seconde partie, on considère des canaux dans lesquels circule un fluide. On montre que la forme d'un canal pour lequel la configuration de l'écoulement et la température varient longitudinalement peut être optimisée de telle façon que la résistance globale du conduit soit minimisée. L'optimisation des formes du conduit illustrée ici est conduite avec une des deux contraintes, volume total constant du conduit ou surface totale de la paroi constant.

DER ANSATZ VON ERNST SCHMIDT ZUR RIPPEN-OPTIMIERUNG—ERWEITERUNG AUF RIPPEN MIT VERÄNDERLICHER WÄRMELEITFÄHIGKEIT UND DIE GESTALTUNG VON STRÖMUNGSKANÄLEN

Zusammenfassung—Im ersten Teil der Arbeit werden die intuitiven Schlußfolgerungen von Ernst Schmidt zur Wahl einer optimalen Rippenform für minimalen Materialverbrauch unter Benutzung der Methode der Variationsrechnung analytisch umgesetzt. Selbst wenn die Wärmeleitfähigkeit der Rippe temperaturabhängig ist, ergibt sich als Optimum eine Rippe von unveränderlicher Form, d. h. ein Rohr, dessen Geometrie sich in Längsrichtung nicht ändert. Die Verallgemeinerung wird bei der Ermittlung einer optimalen Rippenform für den Fall verwendet, daß das verwendete Material eine temperaturabhängige Leitfähigkeit aufweist. Im zweiten Teil der Arbeit wird die analytische Verallgemeinerung der Schmidt'schen Schlußfolgerung auf die Gestaltung von Strömungskanälen angewandt. Es wird gezeigt, daß die Gestalt eines Kanals, in dem sich Strömungsform oder Temperatur mit der Lauflänge verändern, so optimiert werden kann, daß der Gesamtströmungswiderstand ein Minimum erreicht. Die Optimierung der beschriebenen Kanalformen wird entweder bei konstantem Kanalvolumen oder bei konstanter Kanalwandoberfläche durchgeführt.

ОПТИМИЗАЦИЯ РЕБРА МЕТОДОМ ЭРНСТА ШМИДТА: РАСПРОСТРАНЕНИЕ МЕТОДА НА СЛУЧАЙ РЕБЕР С ПЕРЕМЕННОЙ ТЕПЛОПРОВОДНОСТЬЮ И ТРУБОПРОВОДЫ

Аннотация—В первой части работы интуитивный метод Эрнста Шмидта для выбора оптимальной формы ребра с минимальной затратой материала сводится к анализу с использованием метода вариационных исчислений. Показано, что даже в том случае, когда теплопроводность ребра есть функция температуры, оптимальная форма единичной "трубки тока" является обычной, т.е. представляет собой трубку, геометрия которой не меняется в зависимости от продольного положения. Это обобщение метода Шмидта используется при отыскании оптимальной формы ребер, выполненных из материалов с теплопроводностью, зависящей от температуры. Во второй части работы аналитическое обобщение метода Шмидта применяется для конструирования трубопроводов. Показано, что форма трубопровола, режим течения или температура в котором изменяется в продольном направлении, может быть оптимизирована так, чтобы общее гидравлическое сопротивление в трубопроводе было сведено к минимуму. Представленная оптимизация форм трубопровода проведена с учетом одного или двух ограничений: постоянный суммарный объем трубопровода или постоянная общая поверхность его стенок.